follows from (3.1).

Thus the corresponding stress field (2.3) for an element $\varphi \in H$ satisfied the equilibrium equations (1.1) and the boundary conditions (1.2) in the generalized sense (3.4). In particular, if $\varphi \in M \subset H$, then Eq. (1.1) and the boundary conditions (1.2) are satisfied in the usual sense for the stresses (2.3).

We note in conclusion that the results obtained are valid for the entire region Ω occupied by the medium, independently of the distribution of rigid and plastic regions. A proof of the uniqueness of the stress field only for those parts of the body in which the deformation rates are different from zero is given in [2].

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PROBLEM OF PURE SHEAR OF A VISCOPLASTIC MEDIUM BETWEEN TWO NONCOAXIAL CIRCULAR CYLINDERS

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The problem of the flow of a viscoplastic material between two noncoaxial circular cylinders is discussed. An approximate solution is found with the help of the iterative method described in [1, 2]. Analytic methods of solving similar problems are discussed in [3-4]. An approximate solution is found in [6, 7] with the use of iterative methods [8].

1. The problem is solved in a cylindrical coordinate system. The axis O_z is directed parallel to the generating lines of the cylinders, the contours of whose transverse cross section are specified by the equations $R_0 = R_0(\varphi)$ and $R_1 = R_1(\varphi)$. The outer cylinder is fixed, and the inner one moves in the positive direction of the axis Oz with velocity v_*. In this case only one velocity component $v_z = v(r, \varphi)$ is different from zero. In the flow under discussion the components of the deformation rate tensor are of the form

$$e_{rr} = e_{\varphi\varphi} = e_{zz} = e_{r\varphi} = 0, \quad e_{rz} = \frac{1}{2}\frac{\partial v}{\partial r}, \quad e_{\varphi z} = \frac{1}{2r}\frac{\partial v}{\partial \varphi}. \tag{1.1}$$

We will write the relation between the components of the stress tensor σ_{ij} and the components of the deformation rate tensor e_{ij} for a viscoplastic medium with the Miesz plasticity condition in the form [9]

$$\sigma_{ij} = \left(\frac{\sqrt{2}k}{\sqrt{e_{kl}e_{kl}}} + 2\mu\right) e_{ij} - p_1\delta_{ij}, \tag{1.2}$$

where p_1 is the hydrostatic pressure, k is the yield point, and μ is the viscosity coefficient. Substituting (1.1) into (1.2), we obtain

$$\sigma_{rr} = \sigma_{\varphi\varphi} = \sigma_{zz} = -p_1, \; \sigma_{r\varphi} = 0,$$
$$\sigma_{rz} = \frac{k+\mu\gamma}{\gamma}\frac{\partial v}{\partial r}, \quad \sigma_{\varphi z} = \frac{k+\mu\gamma}{r\gamma}\frac{\partial v}{\partial \varphi}, \quad \gamma = \sqrt{\left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v}{\partial \varphi}\right)^2}. \tag{1.3}$$

We write the equilibrium equations

$$\frac{\partial p_1}{\partial r} = \frac{\partial p_1}{\partial \varphi} = 0, \quad \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\varphi z}}{\partial \varphi} + \frac{\sigma_{rz}}{r} - \frac{\partial p_1}{\partial z} = 0. \tag{1.4}$$

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It follows from (1.4) that $\partial p_1/\partial z = p = \text{const}$. Let us convert to dimensionless quantities. In order to do this, we will refer the stresses to the yield point k, linear dimensions to the quantity $h = \max_{\varphi} R_0(\varphi)$, v_* in the pressure p to the quantity k/h.

After substitution of (1.3) into (1.4) we arrive at the equilibrium equation in dimensionless form

$$H\left(v_{rr} + \frac{v_{\varphi\varphi}}{r^2} + \frac{v_r}{r}\right) + \frac{v_{rr}v_\varphi^2 - 2v_r v_\varphi v_{r\varphi} + v_{\varphi\varphi}v_r^2 + 2v_r \frac{v_\varphi^2}{r} + v_r^3 r}{r^2\left(v_r^2 + \frac{v_\varphi^2}{r^2}\right)^{3/2}} - p = 0, \tag{1.5}$$

where $H = \mu v_*/kh$; $v_\varphi = \partial v/\partial\varphi$, and $v_r = \partial v/\partial r$.

The boundary conditions in the problem under discussion are the following:

$$v|_{r=R_0(\varphi)} = 1, \quad v|_{r=R_1(\varphi)} = 0. \tag{1.6}$$

2. We will briefly describe the essence of the method proposed in [1, 2] for obtaining an approximate solution of the boundary-value problem (1.5) and (1.6). We will switch in Eq. (1.5) from the variables r, φ to the new variables ξ, φ with the help of the transformation

$$\varphi = \varphi, \ \xi = f(r, \varphi). \tag{2.1}$$

We assume that the transformation (2.1) is selected so that it is possible to neglect in the new variables derivatives of v with respect to φ and mixed derivatives in comparison with the remaining terms. Then we obtain for the quantity v a linear ordinary second-order differential equation with variable coefficients

$$H\left(\xi_r^2 + \frac{\xi_\varphi^2}{r^2}\right)v_{\xi\xi} + H\left(\xi_{rr} + \frac{\xi_{\varphi\varphi}}{r^2} + \frac{\xi_r}{r}\right)v_\xi + \frac{\xi_{rr}\xi_\varphi^2 - 2\xi_r\xi_\varphi\xi_{r\varphi} + \xi_{\varphi\varphi}\xi_r^2 + 2\xi_r\frac{\xi_\varphi^2}{r} + \xi_r^3 r}{r^2\left(\xi_r^2 + \frac{\xi_\varphi^2}{r^2}\right)^{3/2}} \operatorname{sgn}(v_\xi) - p = 0, \tag{2.2}$$

which φ enters as a parameter. Let us also assume that the transformation (2.1) is such that $\operatorname{sgn}(v_\xi) = 1$ and the boundary conditions are of the form

$$v|_{\xi=0} = 0, \ v|_{\xi=1} = 1. \tag{2.3}$$

Having solved the boundary-value problem (2.2) and (2.3), we find the first approximate solution $v = v^{(1)} = \mathcal{F}(\xi, \varphi)$. Let us substitute here the expression for ξ from (2.1)

$$v^{(1)} = \mathcal{F}(f(r, \varphi), \varphi) = f_1(r, \varphi). \tag{2.4}$$

Now denoting $v^{(1)}$ by $\xi^{(1)}$ in (2.4), we solve the problem (2.2) and (2.3) with the replacement of variables $\varphi = \varphi$, and $\xi^{(1)} = f_1(r, \varphi)$ in order to find the second iteration. We find the subsequent iterations similarly.

3. We will apply this method to solve the problem (1.5) and (1.6) in the case in which $p = 0$; the contours of the transverse cross section of the cylinders are specified in the form

$$R_0(\varphi) = 1, \ R_1(\varphi) = -\varepsilon\cos(\varphi) + \sqrt{R^2 - \varepsilon^2\sin^2(\varphi)}, \tag{3.1}$$

where ε is the distance between the axes of the cylinders and R is the dimensionless radius of the outer cylinder. The null approximation was taken to be

$$\xi^0 = \frac{r - R_0}{H} + 1 - \frac{R_1 - R_0 + H}{H\ln\left(\frac{R_1}{R_0}\right)}\ln\left(\frac{r}{R_0}\right). \tag{3.2}$$

Equation (3.2) corresponds to the exact solution for $R_1 = \text{const}$ and $R_0 = \text{const}$. In the case (3.1) $v(r, \varphi)$ is an even function of φ.

Therefore, it is sufficient to find a solution in the rectangle $D = (0 \le \xi \le 1, \ 0 \le \varphi \le \pi)$.

The solution in the i-th approximation was found in tabular form

$$v^{(i)} = \{v_{kj}^{(i)}, \quad 0 \leqslant k \leqslant N_0, \ 0 \leqslant j \leqslant N_1\} \tag{3.3}$$

at the nodes of a grid $\omega = \omega_\xi \times \omega_\varphi = \{(\xi_k, \varphi_i), \ \xi_k = kh_0, \ 0 \le k \le N_0, \ \varphi_i = ih_1, \ 0 \le i \le N_1\}$, $h_1 = \pi/N_1$, $h_0 = 1/N_0$, by solving the problem (2.2) and (2.3) with fixed φ_i on the basis of a procedure of reducing it to two Cauchy problems [10].

For fixed φ_i the data of the tables (3.3) were converted with the help of interpolation to the values of the table

$$\xi^0 = \{\xi^0_{kj}, \quad 0 \leqslant k \leqslant N_0, \; 0 \leqslant j \leqslant N_1\}, \tag{3.4}$$

defined at the nodes of the grid $\Omega = \Omega_0 \times \omega_\varphi = \{(v_k^{(i)}, \varphi_l), v_k^{(i)} = kh_0, 0 \le k \le N_0, \varphi_l = lh_1, 0 \le l \le N_1\}$, where $\Omega_0 = \{v_k^{(i)} = kh_0, k = 0, 1, \ldots, N_0, h_0 = 1/N_0\}$. In order to obtain the subsequent approximations it is necessary to differentiate the solution obtained at the preceding step of the iterative process. To this end, the solution in the form of the table (3.4) was replaced by a solution in the form

$$\xi^0 = \sum_{k=0}^{M} a_k (v^{(i)}) \cos (k\varphi), \quad M \leqslant N_1. \tag{3.5}$$

In order to find the coefficients $a_k(v^{(i)})$, the values of the table (3.4) were smoothed by trigonometric polynomials according to the method of least squares with respect to the variable φ with fixed values of $v^{(i)}$ at the nodes of the grid Ω_0. In this connection a table of values was obtained at the nodes of this grid for each a_k, from which an interpolative polynomial is constructed.

The most stable convergence is observed if one successively assigns the values 1, 2,... to M and achieves convergence of the iterative process for each specified M. The application at the nodes of the grid Ω of the Aitken–Steffensen transformation (δ^2-transformation) often gives a large effect [11].

The problem of finding values of H_*, where H_* is the critical value for H corresponding to the onset of the origin of a stagnation zone, depending on the value of the quantity ε, was also posed in the research. When $H = H_*$ at the stagnation point of the stagnation zone $X_* = \{r = R + \varepsilon, \varphi = \pi\}$, we have $v_r^2 + v_\varphi^2/r^2 = 0$, and $v_r < 0$ at all the rest of the points of the flow [9]. These conditions should be satisfied for the approximate solution, i.e., $v_r^{(i)}(X_*) = 0$. Then the Jacobian of the transformation (2.1) vanishes at the point X_*, and Eq. (2.2) has a singularity. Due to this fact some changes are introduced into the solution procedure. In the problem (2.2) and (2.3) we make a replacement of variables which is the inverse of the replacement (2.1), $\varphi = \varphi$, $r = f^{-1}(\xi, r)$, and we obtain the following boundary-value problem:

$$v_{rr} + \left(\frac{\xi_{rr} + \frac{\xi_{\varphi\varphi}}{r^2} + \frac{\xi_r}{r}}{\xi_r^2 + \frac{\xi_\varphi^2}{r^2}} \xi_r - \frac{\xi_{rr}}{\xi_r} \right) v_r + \frac{\xi_{rr}\xi_\varphi^2 - 2\xi_r\xi_\varphi\xi_{r\varphi} + \xi_{\varphi\varphi}\xi_r^2 + 2\xi_r\frac{\xi_\varphi^2}{r} + \xi_r^3 r}{Hr^2\left(\xi_r^2 + \frac{\xi_\varphi^2}{r^2}\right)^{5/2}} \xi_r^2 = 0, \tag{3.6}$$

$$v|_{r=R_0(\varphi)} = 1, \quad v|_{r=R_1(\varphi)} = 0.$$

This problem is equivalent to the problem (2.2) and (2.3). We solve it at all the nodes of the grid ω_φ with the exception of the node $\varphi = \pi$. The differential equation of the problem (3.6) has the form

$$v_{rr} + \frac{1}{r}\left(\frac{\xi_{\varphi\varphi}}{r\xi_r} + 1\right) v_r - \frac{1}{Hr}\left(\frac{\xi_{\varphi\varphi}}{r\xi_r} + 1\right) = 0. \tag{3.7}$$

at $\varphi = \pi$.

Eq. (1.5) has the form

$$H\left(v_{rr} + \frac{v_{\varphi\varphi}}{r^2} + \frac{v_r}{r}\right) - \frac{1}{r}\left(\frac{v_{\varphi\varphi}}{rv_r} + 1\right) = 0.$$

at $\varphi = \pi$. Let us assume that this equation is true for the approximate solution

$$H\left(\xi_{rr} + \frac{\xi_{\varphi\varphi}}{r^2} + \frac{\xi_r}{r}\right) - \frac{1}{r}\left(\frac{\xi_{\varphi\varphi}}{r\xi_r} + 1\right) = 0.$$

Taking account of the latter equation, we transform Eq. (3.7) and solve the problem

$$v_{rr} + H\left(\xi_{rr} + \frac{\xi_{\varphi\varphi}}{r^2} + \frac{\xi_r}{r}\right) v_r - \xi_{rr} - \frac{\xi_{\varphi\varphi}}{r^2} - \frac{\xi_r}{r} = 0, \tag{3.8}$$

$$v|_{r=1} = 1, \; v|_{r=R+\varepsilon} = 0.$$

at $\varphi = \pi$ instead of the problem (3.6).

In this case we use for smoothing the representation of the approximate solution

$$v^{(i)}(r, \varphi) = \xi^0(r, \varphi) + \sum_{k=0}^{M} a_k(t) \cos (k\varphi), \tag{3.9}$$

where $t = [r - R_0(\varphi)]/[R_1(\varphi) - R_0(\varphi)]$, which we find in advance in the form of a table at the nodes of a uniform grid $\Omega_1 = \{(t_k, \varphi_i), t_k = kh_0, 0 \le k \le N_0, \varphi_i = ih_1, 0 \le i \le N_1\}$. The calculations according to the scheme (3.6), (3.8), and (3.9) are more intuitive but less stable than in the case discussed earlier. The problem (3.8) does not have a singularity at $H = H_*$.

We will find the value of H_* in the following way. We select some H_0 such that $H_0 > H_*$, i.e., for $H = H_0$, $\xi_r < 0$ everywhere in the flow region. We find an approximate solution of the problem $v(r, \varphi)$ for $H = H_0$, and using the representation (3.9), we set

$$v_r(X_*) = 0. \tag{3.10}$$

We find H_1 from (3.10) and seek the solution of the problem at $H = H_1$. In this connection the solution obtained for $H = H_0$ is taken as the zeroth approximation, and thus we proceed to the next iteration.

4. For comparison the problem (1.5) and (1.6) is also solved by the small-parameter method. We take the quantity ε as the small parameter and seek the solution in the form

$$v = v_0(r, \varphi) + \varepsilon v_1(r, \varphi) + \ldots . \tag{4.1}$$

All the quantities which enter into (1.5) and (1.6) are expanded in this parameter into a power series, and we group the terms by identical powers of ε. The boundary conditions (1.6) give the following conditions for the functions v_0 and v_1;

$$\left.\begin{matrix} v_0(1, \varphi) = 1, \\ v_0(R, \varphi) = 0 \end{matrix}\right\}, \quad \left.\begin{matrix} v_1(1, \varphi) = 0, \\ v_1(R, \varphi) = \dfrac{\partial v_0}{\partial r}(R, \varphi)\cos(\varphi) \end{matrix}\right\}.$$

Solving the corresponding boundary-value problems for ordinary differential equations, we obtain to an accuracy of ε in the first order

$$v = \frac{r-1}{H} + 1 - \left(\frac{R-1}{H} + 1\right)\frac{\ln r}{\ln R} + \varepsilon\left(c_1 r F\left(1, 1, 3, \frac{r}{E}\right) + c_2 \frac{E-r}{r}\right)\cos(\varphi), \tag{4.2}$$

$$E = \frac{R + H - 1}{\ln R}, \quad F(1, 1, 3, x) = 1 + 2\sum_{k=1}^{\infty} \frac{x^k}{(k+1)(k+2)},$$

$$c_1 = \frac{(E-1)(E-R)}{(E-R)F\left(1, 1, 3, \frac{1}{E}\right) - R^2(E-1)F\left(1, 1, 3, \frac{R}{E}\right)}\frac{1}{H},$$

$$c_2 = \frac{(R-E)F\left(1, 1, 3, \frac{1}{E}\right)}{(E-R)F\left(1, 1, 3, \frac{1}{E}\right) - R^2(E-1)F\left(1, 1, 3, \frac{R}{E}\right)}\frac{1}{H}.$$

Eq. (4.2) gives an approximate solution of the original problem. The condition for the development of a stagnation zone $v_r(X_*) = 0$ leads to an equation for the determination of H_*

$$\frac{R-E}{HR} + \varepsilon\left(\frac{E}{R^2 H} - c_1 F\left(1, 1, 3, \frac{R}{E}\right) - \frac{c_1 R}{E}F'\left(1, 1, 3, \frac{R}{E}\right) + c_2\frac{E}{R^2}\right) = 0,$$

where

$$F'(1, 1, 3, x) = dF(1, 1, 3, x)/dx.$$

Some of the results obtained with $R = 2$, $M = 2$–3, $N_0 = 8$, $N_1 = 12$, and $h_2 = 1/64$ are given in Tables 1–3. The calculations in Tables 1 and 2 were carried out for $H = 10$; v_0 and v_1 are the terms of the expansion in Eq. (4.1); $v^{(1)}$ is the solution corresponding to a single iteration without subsequent smoothing; and v is the limiting value of the solution. The quantities H_*^0 and H_*^1 in Table 3 are found by the small parameter method with one and two terms in the expansion (4.1), respectively. With $\varepsilon = 0.1$ we take (3.2) as the zeroth approximation. Then setting M successively equal to 1 and 2 in (3.5), we make the δ^2 transformation one at a time. We take the solution obtained as the zeroth approximation for $\varepsilon = 0.3$, and we perform a single δ^2 transformation for $M = 2, 3$. Setting $M = 2$ in the solution found, we take it as the zeroth approximation for $\varepsilon = 0.5$ and again perform a single δ^2 transformation for $M = 2, 3$. We similarly obtain a solution for $\varepsilon = 0.7$ and $\varepsilon = 1$. Since it is necessary to perform two simple iterations in order to accomplish the δ^2 transformation, it is sufficient for obtaining the results given in Tables 1 and 2 according to the procedure expounded above to make four simple iterations for each ε.

TABLE 1

v \ $\varphi°$	0	45	90	135	180
v_0	0,45534	0,44147	0,40774	0,37369	0,35950
v_1	0,42039	0,41609	0,40774	0,40241	0,40113
$v^{(1)}$	0,41620	0,41332	0,40641	0,39959	0,39680
v	0,41608	0,41323	0,40641	0,39968	0,39692

TABLE 2

ε		$\varphi°$: 0	45	90	135	180
0,3	v_1	0,4748	0,4531	0,4174	0,4090	0,4145
	v	0,43495	0,42615	0,40535	0,38536	0,37738
0,5		0,4536	0,4385	0,4031	0,3701	0,3574
0,7		0,4722	0,4504	0,3996	0,3537	0,3369
1		—	0,467	0,391	0,326	0,305

TABLE 3

ε	0,1	0,3	0,5	0,7	1
H_*	0,461	0,640	0,863	1,140	1,69
H_*^0	0,386	0,386			
H_*^1	0,472	0,777			

The example considered shows that the iterative method proposed in [1, 2] proves to be sufficiently effective for the solution of the problems of the antiplanar strain of a viscoplastic medium under conditions of pure shear for different values of the parameters which enter into the problem.

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